VARIATIONAL PRINCIPLES FOR NONSTATIONARY HEAT CONDUCTION PROBLEMS

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Integral variational principles are proposed for nonstationary heat conduction problems. These lead to integration of the wave equation of heat conduction or the Fourier heat conduction equation with the initial and boundary conditions.

Many studies [1–11] have been devoted to the variational formulation of nonstationary problems of heat conduction. Most of these studies involve the Biot integrodifferential variational principle. In the Biot principle the variable expression is a function of time since it is represented in the form of definite integrals only with respect to the space coordinates. Accordingly, in the approximate solution of heat conduction problems the Biot principle is used together with the Kantorovich method which reduces the problem to the solution of systems of ordinary differential equations with respect to time, and not with the Ritz method, making it possible to reduce the problem to the solution of systems of algebraic equations.

In this paper we present integral variational principles that permit the solution of nonstationary heat conduction problems by the Ritz method. The mixed problem of heat conduction was previously formulated as a variational problem in [12]. But the natural condition of stationarity of the functional proposed in [12] is not the differential equation of heat conduction but some equivalent integrodifferential equation.

In formulating the variational principle of heat conduction we start from the wave equation of heat conduction recently derived by Kaliski [13]. The hyperbolic type of this equation ensures a finite rate of propagation of the thermal perturbations, which is particularly important in problems in which, apart from thermal effects, it is necessary to take into account the influence of other fields (electromagnetic, elastic).

In the second and third parts of this paper different variational formulations of the Fourier heat conduction equation are presented.

1. Variational formulation of the wave equation of heat conduction. Let a problem of heat transfer theory be given in the following form: find the solution of the heat conduction equation [13]

$$(kT_{,i})_{,i} = \tau\rho c\ddot{T} + \rho c\dot{T} \quad P \in D, \quad t > 0 \tag{1}$$

with boundary conditions

$$T = f(P,\ t) \quad P \in B_1, \quad t > 0, \tag{2}$$

$$kT_{,i} = q(P,\ t) \quad P \in B_2, \quad t > 0 \tag{3}$$

and initial conditions

$$T = \Phi(P) \quad P \in D, \quad t = 0, \tag{4}$$

$$\tau\dot{T} = \tau\Psi(P) \quad P \in D, \quad t = 0. \tag{5}$$

If the relaxation coefficient $\tau = 0$, then Eq. (1) and relations (2)–(5) reduce to the Fourier heat conduction equation and the corresponding boundary and initial conditions.

We will show that problems (1)–(5) can be formulated as a variational problem.

We denote the convolution integrals with respect to time t by

$$A * B = \int_0^{t_1} A(P,\ t)B(P,\ t_1 - t)\,dt \tag{6}$$

and introduce the functional

$$\begin{aligned} I = \frac{1}{2}\int_D \{&kT_{,i} * T_{,i} + \tau\rho c\dot{T} * \dot{T} + \rho c\dot{T} * T + \\ &+ 2\tau\rho c[T(P,\ 0) - \Phi(P)]\,\dot{T}(P,\ t_1) + \\ &+ \rho c[T(P,\ 0) - 2\Phi(P)]\,T(P,\ t_1) - \\ &- 2\tau\rho c\,\Psi(P)\,T(P,\ t_1)\}\,dV - \\ &- \int_{B_1} q * T dS - \int_{B_2} k(T - f) * T_{,i}n_i dS. \end{aligned} \tag{7}$$

We have the following variational principle: for a true temperature distribution satisfying Eq. (1) and conditions (2)–(5) in the time interval $(0, t_1)$ the functional (7) has a stationary value, i.e.,

$$\delta I = 0. \tag{8}$$

In fact, forming the first variation of functional (7) and transforming it using the Ostrogradskii formula, we obtain

$$\begin{aligned} \delta I = \int_D \{&[-(kT_{,i})_{,i} + \rho c\dot{T} + \tau\rho c\ddot{T}] * \delta T + \\ &+ \rho c[T(P,\ 0) - \Phi(P)][\delta T(P,\ t_1) + \tau\delta\dot{T}(P,\ t_1)] + \\ &+ \tau\rho c[\dot{T}(P,\ 0) - \Psi(P)]\,\delta T(P,\ t_1)\}\,dV + \\ &+ \int_{B_1} (kT_{,i} - q) * \delta T dS - \\ &- \int_{B_2} k(T - f) * \delta T_{,i}n_i dS. \end{aligned} \tag{9}$$

Given the fundamental lemma of the calculus of variations, the validity of the above variational principle follows directly from relation (9).

2. Variational formulations of the Fourier heat conduction equation. When $\tau = 0$ functional (7) simplifies to the form

$$I = \frac{1}{2}\int_D \{kT_{,i} * T_{,i} + \rho c\dot{T} * T + \rho c[T(P, 0) - 2\Phi(P)]T(P, t_1)\}\, dV - \int_{B_1} q * T dS - \int_{B_2} k(T - f) * T_{,i} n_i dS. \qquad (10)$$

The stationarity conditions of functional (10) are the equation

$$(kT_{,i})_{,i} = \rho c\dot{T} \quad P \in D, \quad t > 0, \qquad (11)$$

boundary conditions (2), (3) and initial condition (4).

Functional (10) can be reduced to "canonical form":

$$I = \frac{1}{2}\int_D \Big\{-\frac{1}{k} G_i * G_i + \rho c\dot{T} * T - 2G_i * T_{,i} + \rho c[T(P, 0) - 2\Phi(P)]T(P, t_1)\Big\}\, dV - \int_{B_1} q * T dS - \int_{B_2} k(T - f) * G_i n_i dS. \qquad (12)$$

Here, the independent variable functions are T and G_i. Correspondingly, the Euler-Lagrange equations of functional (12) will be

$$\rho c\dot{T} = -G_{i,i}, \qquad (13)$$

$$T_{,i} = -\frac{1}{k} G_i. \qquad (14)$$

Systems (13), (14) are equivalent to Eq. (11).

3. The Biot variational method. Equation (11) can also be represented in the form of the equivalent system

$$\rho cT = -H_{i,i}, \qquad (15)$$

$$T_{,i} = -\frac{1}{k}\dot{H}_i. \qquad (16)$$

The Biot variational equation [2–7] relates precisely to system (15), (16) and has the form

$$\delta \frac{1}{2}\int_D \rho cT^2 dV + \int_D \frac{1}{k}\dot{H}_i \delta H_i = \int_B T \delta H_i n_i dS. \qquad (17)$$

Here, it is assumed that the variable quantities T, H_i satisfy Eq. (15), i.e.,

$$\rho c \delta T = -\delta H_{i,i}. \qquad (18)$$

The idea underlying the construction of the variational equations in the earlier sections of this paper can also be used to formulate the integral variational principle corresponding to integrodifferential equation (17).

To be specific, we will take the boundary and initial conditions for Eqs. (15), (16) in the form

$$T = f(P, t) \quad P \in B, \quad t > 0, \qquad (19)$$

$$T = \Phi(P) \quad P \in D. \qquad (20)$$

The problem (15), (16), (19), (20) can be formulated in the form of the following variational principle: if the temperature T and the heat flow vector H_i are related by Eq. (15) and initial condition (20) is satisfied, then the solution of Eq. (16) with boundary condition (19) is such that the variational equation

$$\delta\Big\{\int_D \Big(\rho cT * T + \frac{1}{k}\dot{H}_i * H_i\Big) dV + 2\int_B f * H_i n_i dS\Big\} = \frac{1}{k}[H_i(P, 0)\delta H_i(P, t_1) - H_i(P, t_1)\delta H_i(P, 0)] \qquad (21)$$

is satisfied.

In order to demonstrate the validity of this principle, we used relation (18) and the Ostrogradskii formula to reduce Eq. (21) to the form

$$2\int_D \Big(T_{,i} + \frac{1}{k}\dot{H}_i\Big) * \delta H_i dV - 2\int_B (T - f) * \delta H_i n_i \, dS = 0. \qquad (22)$$

Equation (16) and boundary condition (19) follow directly from relation (22).

NOTATION

T is the temperature; $(\ )_{,i}$ is the partial derivative with respect to the space coordinate x_i, i.e., $T_{,i} = \partial T/\partial x_i$ $(i = 1, 2, 3)$; $(\dot{\ })$ is the partial derivative with respect to time, i.e., $\dot{T} = \partial T/\partial t$; k is the thermal conductivity; τ is the thermal relaxation coefficient; ρ is the density; c is the specific heat of material; P is the point with coordinates x_i; D is the region occupied by body; B is the boundary surface of body; B_1 is the part of boundary surface where the temperature distribution is given; B_2 is the rest of surface where heat supply is given; n_i are the components of the unit vector along the exterior normal to boundary surface; f is the given temperature distribution; q is the given heat supply; Φ is the given initial temperature distribution; Ψ is the given initial temperature "rate" distribution; H_i is the heat flux vector.

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